Math 255B Lecture 18 Notes

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1 Introduction to Spectral Theory of Unbounded Operators

1.1 The Dirichlet realization of a 2nd order elliptic operator

Let $q : D(q) \to \mathbb{R}$ be a nonnegative, symmetric, densely defined, closed quadratic form. last time, we saw that there is a unique self-adjoint operator $\mathscr{A}$ with $D(\mathscr{A}) \subseteq D(q)$, $q(u,v) = \langle u, \mathscr{A} v\rangle$ for $u \in D(q)$ and $v \in D(\mathscr{A})$. We have

$$D(\mathscr{A}) = \{v \in D(q) : \exists f \in H \text{ s.t. } q(u,v) = \langle u, f\rangle \ \forall u \in D(q)\}.$$

Example 1.1 (Dirichlet realization of a 2nd order elliptic operator)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $\Omega \ni x \mapsto (a_{j,k}(x)) \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ with $a_{j,k} = a_{k,j}$ with $a_{j,k} \in L^\infty(\Omega)$. Assume the **ellipticity condition**:

$$\exists c > 0 \text{ such that} \qquad \sum_{j,k=1}^n a_{j,k}(x)\xi_j\xi_k \geq c|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Let

$$q(u) = \int_\Omega \sum_{j,k=1}^n a_{j,k}(x) \frac{\partial u}{\partial x_j} \overline{\frac{\partial u}{\partial x_k}}\, dx,$$

where $D(q) = H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) : \partial_{x_j} u \in L^2\}$. Then q is closed: $q(u) + \|u\|_{L^2}^2$ is equivalent to $\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2$ on $H_0^1(\Omega)$.

Associated to q is a self-adjoint operator $\mathscr{A}$ with

$$\begin{aligned} D(\mathscr{A}) &= \{u \in H_0^1 : \exists f \in L^2 \text{s.t. } q(u,v) = \langle f, v\rangle \ \forall v \in H_0^1(\Omega)\} \\ &= \{u \in H_0^1 : \exists f \in L^2 \text{s.t. } q(u,v) = \langle f, v\rangle \ \forall v \in H_0^1(\Omega)\} \end{aligned}$$

Rewrite this condition: $\int \sum a_{j,k} \frac{\partial u}{\partial x_j} \overline{\frac{\partial v}{\partial x_k}} = \int f \overline{v} \,\forall v \iff \sum_{j,k=1}^n \partial_{x_j}(a_{j,k}(x) \frac{\partial u}{\partial x_k}) \in L^2(\Omega)$.

$$= \left\{u \in H_0^1 : \sum \partial_{x_j}\left(a_{j,k} \frac{\partial u}{\partial x_k}\right) \in L^2\right\}.$$

$\mathscr{A}$ is given by

$$\mathscr{A}u = -\sum \partial_{x_j}\left(a_{j,k}\frac{\partial u}{\partial x_k}\right), \qquad u \in D(\mathscr{A}).$$

The operator $\mathscr{A}$ is the **Dirichlet realization** of $-\sum_{j,k=1}^{n}\partial_{x_j}(a_{j,k}\frac{\partial u}{\partial x_k})$.

We have

$$\langle \mathscr{A}u, u\rangle = q(u) = \int \sum a_{j,k}\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}} \geq c\|\nabla u\|_{L^2}^2 \geq c'\|u\|_{L^2}^2,$$

where the last inequality is **Poincaré's inequality**. So we get $\mathscr{A} : D(\mathscr{A}) \to L^2(\Omega)$ is bijective: $\mathscr{A}$ is injective, $\operatorname{im} A$ is closed (by Cauchy-Schwarz), and $(\operatorname{im}\mathscr{A})^\perp = \ker\mathscr{A} = \{0\}$.

Remark 1.1. When $(a_{j,k}) = 1$, the corresponding operator is $\mathscr{A} = -\Delta_D$ with $(-\Delta_D) = \{u \in H_0^1 : \Delta u \in L^2\}$. One can show that if $\partial\Omega \in C^2$, then $D(-\Delta_D) = (H_0^1 \cap H^2)(\Omega)$.

Let's look at spectral properties of $\mathscr{A}$.

$\mathscr{A}^{-1}$ is bounded: $L^2(\Omega) \to L^2(\Omega)$, and it is also bounded $L^2(\Omega) \to D(\mathscr{A})$ (equipped with the graph norm). There is a natural embedding $D(\mathscr{A}) \to H_0^1(\Omega) \to L^2(\Omega)$, where the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact (Rellich compactness theorem). Thus, $\mathscr{A}^{-1}$ is compact, and self-adjoint on $L^2(\Omega)$. If $\lambda_1 \geq \lambda_2 \geq \cdots \to 0$ are the nonvanishing eigenvalues of $\mathscr{A}^{-1}$ (each eigenvalue repeated according to its multiplicity), then let $e_n \in L^2(\Omega)$ be the corresponding eigenfunctions: $(\mathscr{A}^{-1} - \lambda_n)e_n = 0$. The e_n form an orthonormal basis of $L^2(\Omega)$; moreover, $e_n \in D(\mathscr{A})$, and $(\mathscr{A} - 1/\lambda_n)e_n = 0$.

Now for $z \in \mathbb{C}$, $\mathscr{A} - z : D(\mathscr{A}) \to L^2(\Omega)$ is invertible if and only if $z \neq 1/\lambda_n$ for all n:

$$\mathscr{A} - z = \underbrace{(1 - z\mathscr{A}^{-1})}_{\text{invertible iff injective}}\mathscr{A},$$

and the first term is injective iff $z \neq 1/\lambda_n$ for all n. We can conclude that the **spectrum** of $\mathscr{A}$ is given by the eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \to \infty$ with $(\mathscr{A} - \mu_n)e_n = 0$ and e_ns forming an orthonormal basis for $L^2(\Omega)$.

1.2 Spectrum and resolvent

Definition 1.1. Let $T : D(T) \to H$ be closed and densely defined. We say that for $\lambda \in \mathbb{C}$, $\lambda \notin \operatorname{Spec}(T)$ if and only if $T - \lambda : D(T) \to H$ is bijective. The complement of $\operatorname{Spec}(T)$ is the **resolvent set** of T. When $\lambda \notin \operatorname{Spec}(T)$, we let $R(\lambda) = (T - \lambda)^{-1}$ be the **resolvent** of T.

Since T is closed, $R(\lambda)$ is closed. By the closed graph theorem, $R(\lambda) \in \mathcal{L}(H, H)$.

Proposition 1.1. *The resolvent set $\rho(T) \subseteq \mathbb{C}$ is open, and $\rho(T) \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(H,H)$ is holomorphic.*

Next time we will prove this. We are working towards a development of the spectral theorem for unbounded operators.